TRANSVERSELY PROJECTIVE FOLIATIONS ON SURFACES: EXISTENCE OF NORMAL FORMS AND PRESCIPTION OF THE MONODROMY

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Abstract. We introduce a notion of normal form for transversely projective structures of singular foliations on complex manifolds. Our first main result says that this normal form exists and is unique when ambient space is two-dimensional. From this result one obtains a natural way to produce invariants for transversely projective foliations on surfaces. Our second main result says that on projective surfaces one can construct singular transversely projective foliations with prescribed monodromy.

1. Introduction and Statement of Results

1.1. **Singular Transversely Projective Foliations.** Classically a smooth holomorphic transversely projective foliation on a complex manifold M is a codimension one smooth holomorphic foliation locally induced by holomorphic submersions on $\mathbb{P}^1_{\mathbb{C}}$ and with transitions functions in $\mathrm{PSL}(2,\mathbb{C})$. Among a number of equivalent definitions that can be found in the literature, we are particularly fund of the following one: $\mathcal{F}$ is a **transversely projective foliation** of a complex manifold M if there exists

(1) $\pi : P \to M$ a $\mathbb{P}^1$-bundle over M;
(2) $\mathcal{H}$ a codimension one foliation of P transversal to the fibration π;
(3) $\sigma : M \to P$ a holomorphic section transverse to $\mathcal{H}$;

such that $\mathcal{F} = \sigma^*\mathcal{H}$. The datum $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \to P)$ is the **transversely projective structure** of $\mathcal{F}$. A nice property of this definition is that the isomorphism class of the $\mathbb{P}^1$-bundle P is an invariant canonically attached to the foliation $\mathcal{F}$, whenever $\mathcal{F}$ has a leaf with non-trivial holonomy, cf. [8, page 177, Ex. 3.24.i].

In the holomorphic category the existence of smooth holomorphic foliations imposes strong restrictions on the complex manifold. For instance there exists a complete classification of smooth holomorphic foliation on compact complex surfaces, cf. [3] and references there within. An interesting corollary of this classification is that a rational surface carries a holomorphic foliation if, and only if, it is a Hirzebruch surface and the foliation is a rational fibration.

On the other hand the so called Riccati foliations on compact complex surfaces S , i.e., the foliations which are transversal to a generic fiber of a rational fibration, are examples of foliations which are transversely projective when restricted to the open set of S where the transversality of $\mathcal{F}$ with the rational fibration holds.

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The problem of defining a *good* notion of singular transversely projective foliation on compact complex manifolds naturally emerges. A first idea would be to consider singular holomorphic foliations which are transversely projective on Zariski open subsets. Although natural, the experience shows that such concept is not very manageable: it is too permissive. With an eye on applications one is lead to impose some kind of regularity at infinity. A natural regularity condition was proposed by Scárdua in [15]. Loosely speaking, it is imposed that the *transversely projective structure* is induced by a global meromorphic triple of 1-forms. The naturality of such definition has been confirmed by the recent works of Casale on the extension of Singer's Theorem [4] and of Malgrange on Non-Linear Differential Galois Theory [10, 5].

At this work we will adopt a variant of the above mentioned definition which maintains the geometric flavor of the definition of a smooth transversely projective foliation given at the beginning of the introduction. For us, $\mathcal{F}$ is a **singular transversely projective foliation** if there exists

(1) $\pi : P \to M$ a $\mathbb{P}^1$-bundle over M;
(2) $\mathcal{H}$ a codimension one singular holomorphic foliation of P transverse to the generic fiber of π;
(3) $\sigma : M \dashrightarrow P$ a meromorphic section generically transverse to $\mathcal{H}$;

such that $\mathcal{F} = \sigma^*\mathcal{H}$. Like in the regular case we will call the datum $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$ a **singular transversely projective structure** of $\mathcal{F}$.

A first remark is that unlike in the regular case the isomorphism class of P is not determined by $\mathcal{F}$ even when we suppose that $\mathcal{F}$ is not *singular transversely affine*. In general the $\mathbb{P}^1$-bundle P is unique just up to bimeromorphic bundle transformations. Thus the invariant that we obtain is the bimeromorphism class of P. When M is projective this invariant is rather dull: any two $\mathbb{P}^1$-bundles over M are bimeromorphic.

To remedy this lack of unicity what we need is a

1.2. **Normal form for a singular transversely projective structure.** To a singular transversely projective structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma)$ we associate the following objects on M:

- **the branch locus**, denoted by $\mathrm{Branch}(\mathcal{P})$, is the analytic subset of M formed by the points $p \in M$ such that $\sigma(p)$ is tangent to $\mathcal{H}$;
- **the indeterminacy locus**, denoted by $\mathrm{Ind}(\mathcal{P})$, is the analytic subset of M corresponding to the indeterminacy locus of σ;
- **the polar divisor**, denoted by $(\mathcal{P})_\infty$, is the divisor on M defined by the direct image under π of the tangency divisor of $\mathcal{H}$ and the one-dimensional foliation induced by the fibers of π.

Two transversely projective structures $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma)$ and $\mathcal{P}' = (\pi : P' \to M, \mathcal{H}', \sigma')$ are said to be **bimeromorphically equivalent** if there exists a bimeromorphism $\phi : P \dashrightarrow P'$ such that $\phi^*\mathcal{H}' = \mathcal{H}$ and the diagram

$$\begin{tikzcd} P \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"] & & P' \arrow[dl, "\pi'"'] \\ & M \arrow[ul, dashed, bend left, "\sigma"] \arrow[ur, dashed, bend right, "\sigma'"'] & \end{tikzcd}$$

commutes.

We will say that a singular transversely projective structure $\mathcal{P}$ is in **normal form** when $\operatorname{cod}\operatorname{Branch}(\mathcal{P}) \leq 2$ and the divisor $(\mathcal{P}')_\infty - (\mathcal{P})_\infty$ is effective, i.e. $(\mathcal{P}')_\infty - (\mathcal{P})_\infty \geq 0$, for every projective structure $\mathcal{P}'$ bimeromorphic to $\mathcal{P}$ satisfying $\operatorname{cod}\operatorname{Branch}(\mathcal{P}') \leq 2$.

Theorem 1. *Let $\mathcal{F}$ be a singular transversely projective foliation on a complex surface S. Every transversely projective structure $\mathcal{P}$ of $\mathcal{F}$ is bimeromorphically equivalent to a transversely projective structure in normal form. Moreover this normal form is unique up to $\mathbb{P}^1$-bundle isomorphisms.*

We do not know if a normal form always exists on higher dimensional complex manifolds. Although when a normal forms exists our prove of Theorem 1 shows that it is unique.

From the unicity of the normal we can systematically produce invariants for singular transversely projective foliations on complex surfaces. For singular transversely projective foliations on the projective plane we define the

1.3. **Eccentricity of a Singular Transversely Projective Structure.** Let $\mathcal{P} = (\pi : P \dashrightarrow \mathbb{P}^2, \mathcal{H}, \sigma : \mathbb{P}^2 \dashrightarrow P)$ be a singular transversely projective structure in normal form of a foliation $\mathcal{F}$ of the projective plane $\mathbb{P}^2$. We define the **eccentricity** of $\mathcal{P}$, denoted by $\operatorname{ecc}(\mathcal{P})$, as follows: if $L \subset \mathbb{P}^2$ is a generic line and $P|_L$ is the restriction of the $\mathbb{P}^1$-bundle P to L then we set $\operatorname{ecc}(\mathcal{P})$ as minus the self-intersection in $P|_L$ of $\overline{\sigma(L)}$.

It turns out that the eccentricity of $\mathcal{P}$ can be easily computed once we know the degree of the polar divisor. More precisely we have the

Proposition 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ and $\mathcal{P}$ a singular transversely projective structure for $\mathcal{F}$ in normal form. Then*

$$\operatorname{ecc}(\mathcal{P}) = \deg(\mathcal{P})_\infty - (\deg(\mathcal{F}) + 2)\,.$$

We do not know if it is possible to give upper bounds for $\operatorname{ecc}(\mathcal{P})$ just in function of the degree of $\mathcal{F}$. A positive result on this direction would be relevant for what is nowadays called the Poincaré Problem.

The next result shows that $\operatorname{ecc}(\mathcal{P})$ captures dynamical information about $\mathcal{F}$ in some special cases.

Proposition 2. *Let $\mathcal{F}$ be a quasi-minimal singular transversely projective foliation of $\mathbb{P}^2$ and $\mathcal{P}$ be a transversely projective structure for $\mathcal{F}$ in normal form. If the monodromy representation of $\mathcal{P}$ is not minimal then*

$$\operatorname{ecc}(\mathcal{P}) > 0\,.$$

An immediate corollary is that transversely projective structures in normal form of Hilbert modular foliations on $\mathbb{P}^2$ have positive eccentricity. This follows from Proposition 2 and the well-known facts that these foliations are transversely projective, quasi-minimal and with monodromy contained in $\mathrm{PSL}(2,\mathbb{R})$, cf. [11, Theorem 1].

1.4. **The Monodromy Representation.** A very important invariant of a projective structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$, is the **monodromy representation**. It is the representation of $\pi_1(M \setminus |(\mathcal{P})_\infty|)$ into $\mathrm{PSL}(2,\mathbb{C})$ obtained by lifting paths on $M \setminus |(\mathcal{P})_\infty|$ to the leaves of $\mathcal{H}$.

Given a hypersurface $H \subset M$ and a representation $\rho : \pi_1(M \setminus H) \to \mathrm{PSL}(2,\mathbb{C})$, one might ask if there exists a foliation $\mathcal{F}$ of M with transversely projective structure $\mathcal{P}$ whose monodromy is ρ.

We will show in §5.1 that the answer is in general no: there are local obstructions to solve the *realization problem*.

On the other hand if the ambient is two-dimensional and the representation ρ lifts to a representation $\tilde{\rho} : \pi_1(M \setminus H) \to \mathrm{SL}(2,\mathbb{C})$ then we have the

Theorem 2. *Let S be a projective surface and H a reduced hypersurface on S. If*

$$\rho : \pi_1(S \setminus H) \to \mathrm{PSL}(2,\mathbb{C})$$

is a homomorphism which lifts to a homomorphism $\tilde{\rho} : \pi_1(S \setminus H) \to \mathrm{SL}(2,\mathbb{C})$ then there exists a singular transversely projective foliation $\mathcal{F}$ with a singular transversely projective structure in normal form $\mathcal{P}$ such that

1. *$H - (\mathcal{P})_\infty \geq 0$;*
2. *ρ is the monodromy representation of $\mathcal{P}$;*
3. *If ρ is not solvable then $\mathcal{F}$ admits a unique singular transversely projective structure in normal form.*

We point out that the result (and the proof here presented) holds for higher dimensional projective manifolds if one supposes that H is a normal crossing divisor, cf. §5.2 for details.

2. Generalities

2.1. **A local description of $\mathcal{H}$.** Let $\Delta^n \subset \mathbb{C}^n$ be a polydisc and $\pi : P \to \Delta^n$ be a $\mathbb{P}^1$-bundle. Since the polydisc is a Stein contractible space we can suppose that P is the projectivization of the trivial rank 2 vector bundle over Δ^n and write $\pi(x, [z_1 : z_2]) = x$. If $\mathcal{H}$ is a codimension one foliation of P generically transversal to the fibers of π then $\pi^*\mathcal{H}$ is induced by a 1-form Ω that can be written as

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \beta z_1 \cdot z_2 + \gamma z_2^2\,,$$

where α, β and γ are meromorphic 1-forms on Δ^n. The integrability condition $\Omega \wedge d\Omega = 0$ translates into the relations

$$\begin{cases} d\alpha &= \alpha \wedge \beta \\ d\beta &= 2\alpha \wedge \gamma \\ d\gamma &= \beta \wedge \gamma \end{cases} \tag{1}$$

The divisor of poles of Ω corresponds to the fibers of π that are tangent to $\mathcal{H}$, i.e., if $\mathcal{C}$ denotes the 1-dimension foliation induced by the fibration π then

$$(\Omega)_\infty = \mathrm{tang}(\mathcal{H}, \mathcal{C})\,.$$

Associated to Ω we have an integrable differential $\mathfrak{sl}(2,\mathbb{C})$-system on the trivial rank 2 vector bundle over Δ^n defined by

$$dZ = A \cdot Z \quad \text{where} \quad A = \begin{pmatrix} -\frac{\beta}{2} & -\gamma \\ \alpha & \frac{\beta}{2} \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

The matrix A can be thought as a meromorphic differential 1-form on Δ^n taking values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and satisfying the integrability condition $dA + A \wedge A = 0$. Darboux's Theorem (see [8], III, 2.8, iv, p.230) asserts that on any simply connected open subset $U \subset \Delta^n \setminus (\Omega)_\infty$ there exists a holomorphic map

$$\Phi : U \to \mathrm{SL}(2,\mathbb{C}) \quad \text{such that} \quad A = \Phi^* M$$

where M is the Maurer-Cartan 1-form on $\mathrm{SL}(2,\mathbb{C})$. Moreover, the map Φ is unique up to a left composition with an element in $\mathrm{SL}(2,\mathbb{C})$. For every $v \in \mathbb{C}^2$ the sections

$$\begin{array}{rcl} \varphi_v : U & \to & U \times \mathbb{C}^2 \\ x & \mapsto & (x, \Phi(x) \cdot v) \end{array}$$

are solutions of the differential system above. It follows that the application

$$\begin{array}{rcl} \phi : U \times \mathbb{P}^1 & \to & U \times \mathbb{P}^1 \\ (x,[z_1,z_2]) & \mapsto & (x,[\Phi(x)(z_1,z_2)])\,, \end{array}$$

conjugates the foliation $\mathcal{H}|_U$ with the one induced by the submersion $U \times \mathbb{P}^1 \to \mathbb{P}^1$.

We have just described $\mathcal{H}$ over the points outside $(\Omega)_\infty$. Now we turn our attention to

2.2. **The behaviour of $\mathcal{H}$ over a generic point of $(\Omega)_\infty$.** Let W be an analytic subset of the support of $(\Omega)_\infty$. We will set $S(W)$ as

$$S(W) = \pi^{-1}(W) \cap \mathrm{sing}(\mathcal{H})\,.$$

We will start by analyzing $\mathcal{H}$ over the irreducible components H of $(\Omega)_\infty$ for which $\pi^{-1}(H)$ is $\mathcal{H}$-invariant.

Lemma 2.1. *Let H be an irreducible component of the support of $(\Omega)_\infty$ and*

$$V = \{p \in H \text{ such that } \pi^{-1}(p) \subsetneq \mathrm{sing}(\mathcal{H}) \text{ and } H \text{ is smooth at } p\}.$$

If $\pi^{-1}(H)$ is $\mathcal{H}$-invariant then for every $p \in V$ there exists a neighborhood U of p, a map $\varphi : (U, H\cap U) \to (\mathbb{C},0)$ and a Riccati foliation $\mathcal{R}$ on $(\mathbb{C},0)\times\mathbb{P}^1$ such that

$$\mathcal{H} = \varphi^*\mathcal{R}.$$

In particular $\pi_{|S(V)} : S(V) \to V$ is an étale covering of V of degree 1 or 2.

Proof. Since $\mathrm{cod}\,\mathrm{sing}(\mathcal{H}) \le 2$ then V is a dense open subset of H.

Let $p \in V$ and $F \in \mathcal{O}_{\Delta^n,p}$ be a local equation around for the poles of Ω. Since $p \in V$ at least one of the holomorphic 1-forms $F\alpha, F\beta, F\gamma$ is non-zero at p. After applying a change of coordinates of the form

$$(x,[z_1 : z_2]) \mapsto (x,[a_{11}z_1 + a_{12}z_2 : a_{21}z_1 + a_{22}z_2])$$

where

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right) \in \mathrm{GL}(2,\mathbb{C}),$$

we can assume that $F\alpha, F\beta$ and $F\gamma$ are non-zero at p.

From the relation $d\alpha = \alpha\wedge\beta$ we promptly see that the holomorphic 1-form $F\alpha$ besides being non-singular is also integrable. It follows from Frobenius integrability Theorem and the $\mathcal{H}$-invariance of H that there exist a local system of coordinates $(x, y_2, \ldots, y_n) : U \to \mathbb{C}^n$ where p is the origin of $\mathbb{C}^n$, $F = x^n$ for a suitable $n \in \mathbb{N}$ and $F\alpha = h_0 dx$ for some $h_0 \in \mathcal{O}^*_{\Delta^n,p}$.

Again from the relation $d\alpha = \alpha \wedge \beta$ and the fact that $F\beta(p) = (x^n\beta)(p) \neq 0$ it follows that there exists $h_1 \in \mathcal{O}^*_{\Delta^n,p}$ such that

$$\beta = -\frac{dh_0}{h_0} + h_1\frac{dx}{x^n}.$$

After performing the holomorphic change of variables

$$(x,[z_1 : z_2]) \mapsto (x,[h_0 z_1 : z_2 + (1/2)h_1 z_1])$$

we can suppose that $(\alpha, \beta) = (\frac{dx}{x^n}, 0)$.

The conditions $d\beta = 2\alpha \wedge \gamma$ and $d\gamma = \beta \wedge \gamma$ imply that γ depends only on x: $\gamma = b(x)\frac{dx}{x^n}$, with b holomorphic. Note that on this new coordinate system we can no longer suppose that $F\gamma(p) = x^n\gamma(p) \neq 0$.

Thus on this new coordinate system

$$\Omega = z_1 dz_2 - z_2 dz_1 + z_1^2 \frac{dx}{x^n} + z_2^2 b(x)\frac{dx}{x^n}\,.$$

It follows that on $\pi^{-1}(q)$, $q \in V$, we have one or two singularities of $\mathcal{H}$: one when $b(0) = 0$ and two otherwise. $\square$

A word about the terminology: Further on when we refer to the **transverse type** of an irreducible curve of singularities we will be making reference to the type of singularity of the associated Riccati equation given by the above proposition.

Let us now analyze $\mathcal{H}$ over the irreducible components H of $(\Omega)_\infty$ for which $\pi^{-1}(H)$ is not $\mathcal{H}$-invariant. In the notation of lemma 2.1 we have the

Lemma 2.2. *If $\pi^{-1}(H)$ is not $\mathcal{H}$-invariant then $\pi_{|S(V)} : S(V) \to V$ admits an unique holomorphic section.*

Proof. Let $p \in V$ be an arbitrary point. Without loss of generality we can assume that $F\alpha, F\beta$ and $F\gamma$ are non-zero at p and that H is not invariant by the foliation induced by α, cf. proof of lemma 2.1.

Assume also that $\ker \alpha(p)$ is transverse to H. Thus there exists a suitable local coordinate system $(x, y, y_3, \ldots, y_n) : U \to \mathbb{C}^n$ where p is the origin, $F = x^n$ for some $n \in \mathbb{N}$ and $F\alpha = h_0 dy$ for some $h_0 \in \mathcal{O}^*_{\Delta^n,p}$.

The condition $d\alpha = \alpha \wedge \beta$ implies that $\beta = n\frac{dx}{x} + h_1 \cdot \alpha$ with h_1 meromorphic at p. Since $F\beta = x^n\beta$ is holomorphic and does not vanish at p the same holds for h_1, i.e., $h_1 \in \mathcal{O}^*_{\Delta^n,p}$. Thus if we apply the holomorphic change of coordinates

$$((x, y, y_3, \ldots, y_n), [z_1 : z_2] \mapsto ((x, y, y_3, \ldots, y_n), [z_1 : h_0 \cdot z_2 + h_1 \cdot z_1])$$

we have $d\beta = 0$.

Combining $0 = d\beta = 2\alpha \wedge \gamma$ with $d\gamma = \beta \wedge \gamma$ we deduce that $\gamma = x^n h_3(y)\alpha$ for some meromorphic function h_3. Since γ has poles contained in $H = \{x = 0\}$, h_3 is in fact holomorphic and consequently $\mathcal{H}$ is induced by the 1-form

$$x^n(z_1 dz_2 - z_2 dz_1) + (dy)z_1^2 + (x^n h_3(y)dy)z_2^2\,. \qquad (2)$$

It is now clear that the singular set of $\mathcal{H}$ is given by $\{x = 0\} \cap \{z_1 = 0\}$. Thus there exists an open subset $V_0 \subset V$ for which $S(V_0)$ is isomorphic to V_0. Since $S(V)$ does not contain fibers of $\pi_{|S(V)}$ this is sufficient to prove the lemma. $\square$

Remark 2.3. These irreducible components of $(\Omega)_\infty$ are a kind of *fake* or *apparent* singular set for the transversely projective structures. More precisely, after the fibred birational change of coordinates

$$((x, y, y_3, \ldots, y_n), [z_1 : z_2] \mapsto ((x, y, y_3, \ldots, y_n), [z_1 : x^n z_2])$$

the foliation induced by (2) is completely transversal to the fibres of the $\mathbb{P}^1$-fibration and has a product structure as in the case H is $\mathcal{H}$-invariant.

2.3. **Elementary Transformations.** Still in the local setup, let H be a smooth and irreducible component of the support of $(\Omega)_\infty$ and let $S \subset \pi^{-1}(H)$ be a holomorphic section of the restriction of the $\mathbb{P}^1$-bundle over M. An **elementary transformation** $\mathrm{elm}_S : \Delta^n \times \mathbb{P}^1 \dashrightarrow \Delta^n \times \mathbb{P}^1$ with center in S can be described as follows: first we blow-up S on M and then we contract the strict transform of $\pi^{-1}(H)$. If $F = 0$ is a reduced equation of H and S is the intersection of $H \times \mathbb{P}^1$ with the hypersurface $z_2 = 0$ then elm_S can be explicitly written as

$$\begin{aligned} \mathrm{elm}_S : \Delta^n \times \mathbb{P}^1 &\dashrightarrow \Delta^n \times \mathbb{P}^1 \\ (x, [z_1 : z_2]) &\mapsto (x, [F(x)z_1 : z_2]) \end{aligned}$$

modulo $\mathbb{P}^1$-bundle isomorphisms on the source and the target.

We are interested in describing the foliation $(\mathrm{elm}_S)_*\mathcal{H} = (\mathrm{elm}_S^{-1})^*\mathcal{H}$. More specifically we want to understand how the divisors $\mathrm{tang}(\mathcal{H}, \mathcal{C})$ and $\mathrm{tang}(e_*\mathcal{H}, \mathcal{C})$ are related, where $\mathcal{C}$ denotes the one dimension foliation induced by the fibers to $\Delta^n \times \mathbb{P}^1 \to \Delta^n$. We point out that the analysis we will now carry on can be found in the case $n = 1$ in [3, pages 53–56]. The arguments that we will use are essentially the same. We decided to include them here thinking on readers' convenience.

Let k be the order of $(\Omega)_\infty$ along H. Since $\mathrm{elm}_S^{-1}(x, [z_1 : z_2]) = (x, [z_1 : F(x)z_2])$ it follows that

$$(\mathrm{elm}_S^{-1})^*\Omega = F(z_1dz_2 - z_2dz_1) + \alpha z_1^2 + F\left(\beta + \frac{dF}{F}\right) z_1 \cdot z_2 + F^2\gamma z_2^2\,.$$

Thus the foliation $(\mathrm{elm}_S)_*\mathcal{H}$ is induced by the meromorphic 1-form

$$\widetilde{\Omega} = (z_1dz_2 - z_2dz_1) + \frac{\alpha}{F} z_1^2 + \left(\beta + \frac{dF}{F}\right) z_1 \cdot z_2 + F\gamma z_2^2$$

In order to describe $(\Omega)_\infty$ we will consider three mutually exclusive cases:

(1) $\underline{S \text{ is not contained in } \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})}$: This equivalent to say that $F^k\alpha$ is not identically zero when restricted to H. Therefore

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty + H\,.$$

(2) $\underline{S \subset \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H}) \text{ but } S \text{ is not equal to } \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})}$: This corresponds to $(F^k\alpha)_{|H} \equiv 0$ while $(F^k\beta)_{|H} \not\equiv 0$. If $k \geq 2$ then it follows that

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty\,.$$

When $k = 1$ we have two possible behaviors

$$(\widetilde{\Omega})_\infty = \begin{cases} (\Omega)_\infty - H & \text{when } \beta + \frac{dF}{F} \text{ is holomorphic.} \\ (\Omega)_\infty & \text{otherwise .} \end{cases}$$

(3) $\underline{S \text{ is equal to } \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})}$: Here $(F^k\alpha)|_H \equiv (F^k\beta)|_H \equiv 0$ while $(F^k\gamma)_{|H} \not\equiv 0$.When $k = 1$ it follows that

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty\,.$$

When $k \geq 2$, if we set k' as the smallest positive integer for which $(F^{k'}\alpha)|_H \equiv (F^{k'+1}\beta)|_H \equiv 0$ then

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty - (k - k')H.$$

In the global setup the picture is essentially the same, i.e., if $\pi : P \to M$ is a $\mathbb{P}^1$-bundle over M, H is a smooth hypersurface on M and $s : M \to P|_H$ is a holomorphic section then we build up a new $\mathbb{P}^1$-bundle by blowing up the image of s in P and contracting the strict transform $\pi^{-1}(H)$. The local analysis just made can be applied, as it is, on the global setup.

In the case P is the projectivization of a rank 2 holomorphic vector bundle over a complex manifold M then the elementary transformations just described are projectivizations of the so called elementary modifications, see [7, pages 41–42].

3. Existence and Unicity of the Normal Form

3.1. **Existence of a Normal Form I: A Particular Case.** Let $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : S \dashrightarrow P)$ be a transversely projective structure for a foliation $\mathcal{F}$ on a complex surface S. We will now prove the existence of a normal form for $\mathcal{P}$ under the additional assumptions that the irreducible components of the support of $(\Omega)_\infty$ and the codimension one irreducible components of Branch($\mathcal{P}$) are smooth.

Let H be an irreducible component of $(\mathcal{P})_\infty$ of multiplicity $k(H)$ and, as in §2.2, let $S(H)$ be given by $S(H) = \pi^{-1}(H) \cap \operatorname{sing}(\mathcal{H})$. Thus (see lemmata 2.1 and 2.2) $S(H)$ is an analytic subset of $\pi^{-1}(H)$ formed by a finite union of fibers together with a one or two-valued holomorphic section s of $P|_H$. Note that to assure that s is in fact holomorphic, and not just meromorphic, we have used that H is a curve, i.e., we have used that S is a surface.

If s is two-valued and $k(H) > 1$ then the elementary transformation centered in any of the branches of s (we are, of course, restring to a simply-connected open set where s does not ramifies) will not change the order of poles of Ω), i.e., the order of poles is already minimal. This follows from the fact that the transverse type of s is reduced, it is in fact (cf. [3, page 54]) a saddle-node.

If s is two-valued and $k(H) = 1$ then s does not ramifies. In fact, if the quotient of eigenvalues along one of the branches of s is λ then, by Camacho-Sad index theorem, the other branch will have quotient of eigenvalues equal to $-\lambda$. Thus ramification of s leads to absurdity $\lambda \neq 0$ and $\lambda = -\lambda$. If the quotient of eigenvalues of the branches of s are not integers then we are in a minimal situation. On the contrary if the quotient of eigenvalues of one of the branches of s, say s_+, is a positive integer, say λ_+, then by an elementary transformation centered at s_+ we will obtain two new sections of singularities one of them with transverse type $\lambda_+ - 1$. After λ_+ successive elementary transformations we will arrive at a transversely projective structure, still denoted by $\mathcal{P}$, where H does not belong to the support of $(\mathcal{P})_\infty$ (linearizable transverse type) or s is one-valued(Poincaré-Dulac transverse type).

If s is one-valued, $k(H) = 1$ and H is $\mathcal{H}$-invariant then an elementary transformation centered in s will either transform $\mathcal{H}$ to a foliation with $k(H) = 1$ but now with s two-valued. It changes the transverse type from saddle-node (with weak separatrix in the direction of the fibration) to Poincaré-Dulac. The important fact is that it does not changes $k(H)$.

If s is one-valued, $k(H) > 1$ and H is $\mathcal{H}$-invariant then we have two possibilities. The first is when the transverse type is degenerated. An elementary transformation centered in s will drop the multiplicity of H on $(\mathcal{P})_\infty$. The second possibility is when the transverse type is nilpotent. On this last case the multiplicity is stable by elementary transformations, cf. [3, page 55-56].

If s is one-valued and H is not $\mathcal{H}$-invariant then an elementary transformation centered in s will decrease $k(H)$ by one, compare with remark 2.3. Of course if $k(H)$ reaches zero then the resulting foliation is smooth over a generic point of H.

In resume after applying a finite number of elementary transformations we arrive at a projective structure, still denoted by $\mathcal{P}$, for which $(\mathcal{P})_\infty$ has minimal multiplicity in the same bimeromorphic equivalence class. Note also that no codimension one components have been added to $|(\mathcal{P})_\infty| \cup \mathrm{Branch}(\mathcal{P})$ along the process.

Of course there are distinct biholomorphic equivalence class of projective structure with the same property. To rigidify we have to consider $\mathrm{Branch}(\mathcal{P})$.

Let now H be an irreducible codimension one component of $\mathrm{Branch}(\mathcal{P})$. First suppose that H is contained in the support of $(\mathcal{P})_\infty$. The restriction of σ to $\pi^{-1}(H)$ determines s a natural candidate for center of an elementary transformation. As before, keep the same notation from the projective structure obtained after applying the elementary transformation centered in s. Two things can happen: (1) $\sigma|_H \subset \mathrm{sing}(\mathcal{H})$; or (2) $\sigma|_H \not\subset \mathrm{sing}(\mathcal{H})$. In case (2) we are done. In case (1) we are in a situation no different from the one that we started with. If we iterate the process and keep falling in case (1) we deduce that σ follows the infinitely near singularities of $\mathcal{H}|_H$ and therefore must be an $\mathcal{H}$-invariant hypersurface. Of course this is not the case since in the definition of a transversely projective structure we demand that σ is generically transverse to $\mathcal{H}$.

It remains to consider the case H is not contained in the support of $(\mathcal{P})_\infty$. The elementary transformation centered on s, the restriction of σ to $\pi^{-1}(H)$, yields a projective structure for which we have added H with multiplicity one in $(\mathcal{P})_\infty$. So we have reduced to the case just analyzed: H is contained in the support of $(\mathcal{P})_\infty$.

In resume we have proved the

Proposition 3.1. *Let $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : S \dashrightarrow P)$ be a transversely projective structure for a foliation $\mathcal{F}$ on a complex surface S. Suppose that the irreducible components of the support of $(\Omega)_\infty$ and the codimension one irreducible components of* $\mathrm{Branch}(\mathcal{P})$ *are smooth. Then there exists $\mathcal{P}'$ a transversely projective structure in normal form bimeromorphically equivalent to $\mathcal{P}$.*

Before dealing with the unicity of the normal form we will prove the

3.2. **Existence of a Normal Form II: The General Case.** To prove the existence of a normal form for a general transversely projective structure $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : S \dashrightarrow P)$ for a foliation $\mathcal{F}$ we proceed as follows.

We start by taking an embedded resolution of the support of $(\mathcal{P})_\infty$ and of the codimension one components of $\mathrm{Branch}(\mathcal{P})$, i.e., we take a bimeromorphic morphism $r : \tilde{S} \to S$ such that $r^*|(\mathcal{P})_\infty|$ is a divisor with smooth irreducible components and the codimension one components of $r^*(\mathrm{Branch}(\mathcal{P}))$ are also smooth. We will now work with $\widetilde{\mathcal{P}} = r^*\mathcal{P}$ a transversely projective structure for $\widetilde{\mathcal{F}} = r^*\mathcal{F}$.

Proposition 3.1 implies that there exists a transversely projective structure $\widetilde{\mathcal{P}}'$ in normal form bimeromorphic to $\widetilde{\mathcal{P}}$. If $U = \tilde{S} \setminus D$, where D denotes the exceptional divisor of r, then $\mathcal{P}' = (r|_U)_*(\widetilde{\mathcal{P}}'|_U)$ is a transversely projective structure in normal form for $\mathcal{F}$ defined on the complement of a finite number of points. It follows from Hartog's extension Theorem that to extend $\mathcal{P}'$ it is sufficient to extend the $\mathbb{P}^1$-bundle P'. To conclude we have just to apply the following

Lemma 3.2. *Let $\widetilde{\pi} : \widetilde{P} \to \widetilde{S}$ be a $\mathbb{P}^1$-bundle over a compact complex surface $\widetilde{S}$ and let $r : \widetilde{S} \to S$ be a bimeromorphic morphism with exceptional divisor D. Then there exists a $\mathbb{P}^1$-bundle $\pi : P \to S$ and a map $\phi : \widetilde{P} \to P$ such that $\phi|_{\widetilde{\pi}^{-1}(\widetilde{S}\setminus D)}$ is a $\mathbb{P}^1$-bundle isomorphism.*

Proof. Let $\widetilde{U}$ be a sufficiently small neighborhood of the support of D. We can assume that $U = r(\widetilde{U})$ is a Stein subset of S. Outside $\widetilde{U}$ there is no problem at all: the map r is an biholomorphism when restricted to $\widetilde{S} \setminus \tilde{U}$.

Suppose first that there exists a rank 2 vector bundle $\tilde{E}$ over $\tilde{U}$ such that $\widetilde{P}|_{\tilde{U}} = \mathbb{P}(\tilde{E})$. If $\widetilde{\mathcal{E}}$ denotes the sheaf of sections of $\tilde{E}$, $U = r(\widetilde{U})$ and $p = r(E)$ then Grauert's direct image Theorem assures that $\phi_*\widetilde{\mathcal{E}}$ is a coherent $\mathcal{O}_V$-sheaf. Moreover $\phi_*\widetilde{\mathcal{E}}$ is locally free when restricted to $U \setminus \{p\}$. If $\mathcal{E}^{\vee\vee} = \mathrm{Hom}(\mathrm{Hom}(r_*\mathcal{E}, \mathcal{O}_V), \mathcal{O}_V)$ then $\mathcal{E}^{\vee\vee}$ is a reflexive sheaf. Since we are in dimension two $\mathcal{E}^{\vee\vee}$ is in fact locally free, cf. [7, Proposition 25, page 45]. Thus $\mathcal{E}^{\vee\vee}$ is the sheaf of sections of some rank two vector bundle E. This is sufficient to prove the lemma under the assumption that $\widetilde{P}|_{\tilde{U}} = \mathbb{P}(\tilde{E})$. Now we will show that this is always the case.

The obstruction to a $\mathbb{P}^1$-bundle over $\tilde{U}$ be the projectivization of a rank two vector bundles lies in $\mathrm{H}^2(\tilde{U}, \mathcal{O}^*_{\tilde{U}})$, cf. [1, page 190]. It follows from the exponential sequence that $\mathrm{H}^3(\tilde{U}, \mathbb{Z}) = \mathrm{H}^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$ implies that $\mathrm{H}^2(\tilde{U}, \mathcal{O}^*_{\tilde{U}}) = 0$. But $\tilde{U}$ has the same type of homotopy of a tree of rational curves. Thus $\mathrm{H}^3(\tilde{U}, \mathbb{Z}) = 0$. On the other hand, by [1, Theorem 9.1.(iii), pages 91–92], $\mathrm{H}^2(\tilde{U}, \mathcal{O}_{\tilde{U}}) = \mathrm{H}^2(U, \mathcal{O}_U)$ and this latter group is zero since we have taken U Stein. Consequently we have that $\mathrm{H}^2(\tilde{U}, \mathcal{O}^*_{\tilde{U}}) = 0$ and every $\mathbb{P}^1$-bundle over $\tilde{U}$ is the projectivization of a rank two vector bundle over $\tilde{U}$. □

The examples below show that the lemma 3.2 is no longer true in dimension greater than two.

Example 3.3. Let $f : \mathbb{C}^3 \to \mathbb{C}$ be the function $f(x, y, z) = x^2 + y^2 + z^2$ and consider $\mathcal{F}$ the codimension one foliation induced by the levels of f. If $T\mathcal{F}$ denotes the tangent sheaf of $\mathcal{F}$ then $T\mathcal{F}$ is a rank 2 locally free subsheaf of $T\mathbb{C}^3$ outside the origin of $\mathbb{C}^3$ since at these points f is a local submersion. Nevertheless, at the origin of $\mathbb{C}^3$, $T\mathcal{F}$ is not locally free, i.e., we cannot write

$$df = i_X i_Y dx \wedge dy \wedge dz\,,$$

with X and Y germs of holomorphic vector fields at zero. To see this one has just to observe that, for arbitrary germs of holomorphic vector fields X and Y, the zero set of $i_X i_Y dx \wedge dy \wedge dz$ is either empty or has codimension smaller then two. If $\pi : (\widetilde{\mathbb{C}^3}, D) \to (\mathbb{C}^3, 0)$ denotes the blow-up of the origin of $\mathbb{C}^3$ then, as the reader can check, the tangent sheaf of $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ is locally free everywhere. Now the restriction of π to $\widetilde{\mathbb{C}^3} \setminus |D|$ induces an isomorphism of the $\mathbb{P}^1$-bundles $\mathbb{P}(T\widetilde{\mathcal{F}}|_{\widetilde{\mathbb{C}^3}\setminus|D|})$ and $\mathbb{P}(T\mathcal{F}|_{\mathbb{C}^3\setminus\{0\}})$. Althought the $\mathbb{P}^1$-bundle $\mathbb{P}(T\mathcal{F}|_{\mathbb{C}^3\setminus\{0\}})$ does not extends to a $\mathbb{P}^1$-bundle over $\mathbb{C}^3$. □

A more geometric version of the previous example has been communicated to us by C. Araújo. It has appeared several times in the literature, cf. [2] and references therein. We reproduce it here for the reader's convenience.

Example 3.4. Let p be a point in $\mathbb{P}^3$ and V be the variety of 2-planes in $\mathbb{P}^3$ containing p. Consider the variety $X \subset \mathbb{P}^3 \times V$ defined as

$$X = \{(p, \Pi) \in \mathbb{P}^3 \times V \,|\, p \in \Pi\}.$$

Consider the natural projection $\rho : X \to \mathbb{P}^3$. If $q \neq p$, the fiber over q is the $\mathbb{P}^1$ of planes containing p and q. The fiber over p is naturally identified with V, thus isomorphic to $\mathbb{P}^2$. If $\pi : \widetilde{\mathbb{P}^3} \to \mathbb{P}^3$ is the blow-up of p then we have the following diagram

$$\begin{tikzcd} \widetilde{\mathbb{P}^3} \boxtimes_{\mathbb{P}^3} X \arrow[r] \arrow[d] & X \arrow[d, "\rho"] \\ \widetilde{\mathbb{P}^3} \arrow[r, "\pi"'] & \mathbb{P}^3 \end{tikzcd}$$

The reader can check that the fibered product $\widetilde{\mathbb{P}^3} \boxtimes_{\mathbb{P}^3} X$ is a $\mathbb{P}^1$-bundle over $\widetilde{\mathbb{P}^3}$ and we are in a situation analogous to the previous example. $\square$

To finish the proof of Theorem 1 we have to establish the

3.3. **Unicity of the Normal Form.** Let $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : S \dashrightarrow P)$ and $\mathcal{P}' = (\pi : P' \to S, \mathcal{H}', \sigma : S \dashrightarrow P')$ be two transversely projective structures in normal for the same foliation $\mathcal{F}$ and in the same bimeromorphic equivalence class. Let $\phi : P \dashrightarrow P'$ be a fibered bimeromorphism. We want to show that ϕ is in fact biholomorphic.

Since both $\mathcal{P}$ and $\mathcal{P}'$ are in normal form we have that $(\mathcal{P})_\infty = (\mathcal{P}')_\infty$. Thus for every $p \in S \setminus |(\mathcal{P})_\infty|$ there exists a neighboorhood U of p such that $\mathcal{H}|_{\pi^{-1}(U)}$ and $\mathcal{H}'|_{\pi'^{-1}(U)}$ are smooth foliations transverse to the fibers of π and π', respectively. If ϕ is not holomorphic when restricted to $\pi^{-1}(U)$ then it most contract some fibers of π. This would imply the existence of singular points for $\mathcal{H}'|_{\pi'^{-1}(U)}$ and consequently contradict our assumptions. Thus ϕ is holomorphic over every $p \in S \setminus |(\mathcal{P})_\infty|$.

Suppose now that $p \in |(\mathcal{P})_\infty|$ is a generic point and that Σ_p is germ of curve at p transverse to $|(\mathcal{P})_\infty|$. The restriction of ϕ to $\pi^{-1}(\Sigma)$ (denoted by ϕ_Σ) induces a bimeromorphism of $\mathbb{P}^1$-bundles over Σ. Since Σ has dimension one this bimeromorphism can be written as a composition of elementary transformations. Since p is generic on the fiber $\pi^{-1}(p)$ we have two of three distinguished points: one or two singularities of $\mathcal{H}$ and one point from the section σ. But ϕ_Σ must send these points to the corresponding over the fiber $\pi'^{-1}(p)$. This clearly implies that ϕ_Σ is holomorphic. From the product structure of $\mathcal{H}$ in a neighborhood of p, cf. lemma 2.1 and remark 2.3 after lemma 2.2, it follows that ϕ is holomorphic in a neighborhood of $\pi^{-1}(p)$.

At this point we have already shown that there exists Z, a codimension two subset of S, such that $\phi|_{\pi^{-1}(S\setminus Z)}$ is holomorphic.

Let now $p \in Z$ and U be a neighborhood of p where both P and P' are trivial $\mathbb{P}^1$-bundles. Thus after restricting and taking trivializations of both P and P' we have that $\phi|_{\pi^{-1}(U)}$ can be written as

$$\phi|_{\pi^{-1}(U)}(x, [y_1 : y_2]) = (x, [a(x)y_1 + b(x)y_2 : c(x)y_1 + d(x)y_2]) \,,$$

where a, b, c, d are germs of holomorphic functions. But then the points $x \in U$ where ϕ is not biholomorphic are determined by the equation $(ad - bd)(x) = 0$. Since $(ad - bd)(x)$ is distinct from zero outside the codimension two set Z it is

distinct from zero everywhere. Therefore we conclude that ϕ is fact biholomorphic and in this way conclude the prove of the unicity of the normal form. This also concludes the proof of Theorem 1. □

Remark 3.5. To prove the unicity we have not used that S is a surface. Therefore as long as a normal form exists it is unique no matter the dimension of the ambient manifold.

4. Eccentricity of a Singular Transversely Projective Structure

4.1. **Foliations on the Projective Plane and on $\mathbb{P}^1$-bundles.** The degree of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is defined as the number of tangencies of $\mathcal{F}$ with a general line L on $\mathbb{P}^2$. When $\mathcal{F}$ has degree d it is defined through a global holomorphic section of $\mathrm{T}\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)$, see [3, pages 27–28].

If $\pi : S \to B$ is a $\mathbb{P}^1$-bundle over a projective curve B, $\mathcal{C}$ is the foliation tangent to the fibers of π and $\mathcal{R}$ is a Riccati foliation on S then $\mathcal{R}$ is defined by a global holomorphic section of $\mathrm{T}S \otimes \pi^*(\mathrm{T}^*B) \otimes \mathcal{O}_S(\mathrm{tang}(\mathcal{R}, \mathcal{C}))$ [3, page 57]. If $C \subset S$ is a reduced curve not $\mathcal{R}$-invariant then [3, proposition 2,page 23]

$$\deg\left(\pi^*(\mathrm{T}B) \otimes \mathcal{O}_S(-\mathrm{tang}(\mathcal{R}, \mathcal{C})\right)|_C = C^2 - \mathrm{tang}(\mathcal{R}, C)\,.$$

With these ingredients at hand we are able to obtain

4.2. **A formula for the Eccentricity: Proof of Proposition 1.** Let $L \subset \mathbb{P}^2$ be a generic line and let P_L be the restriction of the $\mathbb{P}^1$-bundle $\pi : P \to \mathbb{P}^2$ to L. On P_L we have $\mathcal{G}$, a Riccati foliation induced by the restriction of $\mathcal{H}$, and a curve C corresponding to $\sigma(L)$. Notice that

$$T\mathcal{G} = (\pi|_L)^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-(\mathcal{P})_\infty)\,.$$

We also point out that the tangencies between $\mathcal{G}$ and C are in direct correspondence with the tangencies between $\mathcal{F}$ and L. Thus

$$\begin{aligned} T\mathcal{G} \cdot C &= C \cdot C - \mathrm{tang}(\mathcal{G}, C) \\ &= -\mathrm{ecc}(\mathcal{P}) - \deg(\mathcal{F})\,. \end{aligned}$$

Combining this with the expression for $T\mathcal{G}$ above we obtain that

$$2 - \deg((\mathcal{P})_\infty) = -\mathrm{ecc}(\mathcal{P}) - \deg(\mathcal{F})\,,$$

and the proposition follows. □

4.3. **Some Examples.** Before proceeding let's see some examples of transversely projective foliations on $\mathbb{P}^2$ and compute theirs eccentricities using proposition 1.

Example 4.1. [Hilbert Modular Foliations on the Projective Plane] In [11] some Hilbert Modular Foliations on the Projective Plane are described. For instance in Theorem 4 of loc. cit. a pair of foliations $\mathcal{H}_2$ and $\mathcal{H}_3$ of degrees 2 and 4 is presented. Both foliations admit transversely projective structures with reduced polar divisor whose support consists of a rational quintic and a line, cf. [6, 11]. For $\mathcal{H}_2$ the eccentricity is equal to $2 = 6 - (2 + 2)$ while for $\mathcal{H}_3$ it is equal to $1 = 6 - (3+2)$. Similarly if one consider the pair of foliations $\mathcal{H}_5$ and $\mathcal{H}_9$ presented in Theorem 2 of loc. cit. then $\mathcal{H}_5$ has eccentricity $8 = 15 - (5 + 2)$ and $\mathcal{H}_9$ has eccentricity $4 = 15 - (9 + 2)$. Since $\mathcal{H}_5$ is birationally equivalent to $\mathcal{H}_9$ and $\mathcal{H}_2$ is birationally equivalent to $\mathcal{H}_3$ these examples show that the eccentricity is not a birational invariant of transversely projective foliations.

Example 4.2. [Riccati Foliations on $\mathbb{P}^2$] Let $p \in \mathbb{P}^2$ be a point and let $\mathcal{F}$ be a degree d foliation for which the singular point p has $l(p) = d$. We recall that $l(p)$ is defined as follows: if $\pi : (\widetilde{\mathbb{P}^2}, E) \to (\mathbb{P}^2, p)$ is the blow-up of p and ω is a local 1-form with codimension two singular set defining $\mathcal{F}$ then $l(p)$ is the vanishing order of $\pi^*\omega$ along the exceptional divisor E.

When $l(p) = d$ it follows from [3, page 28 example 3] that $T\pi^*\mathcal{F} \cdot \overline{L} = 0$, where $\overline{L}$ is the strict transform of a line passing through p. From the discussion in [3, page 50–51] it follows that $\pi^*\mathcal{F}$ is a Riccati foliation.

For a generic degree d foliation $\mathcal{F}$ satisfying $l(p) = d$ we will have $d+1$ invariant lines passing through p and no other invariant algebraic curves. Since $\mathcal{F}$ is Riccati it will have a transversely projective structure with exceptional divisor supported on the $d+1$ $\mathcal{F}$-invariant lines. For generic $\mathcal{F}$ the exceptional divisor will be reduced and with support equal to the union of these lines. In this case we will have that the eccentricity is minus one.

Example 4.3. [Brunella's Very Special Foliation] The very special foliation admits a birational model on $\mathbb{P}^2$ where it is induced by the homogeneous 1-form (cf. [12])

$$\omega = (-y^2z - xz^2 + 2xyz)dx + (3xyz - 3x^2z)dy + (x^2z - 2xy^2 + x^2y)dz \,. \tag{3}$$

It has three invariant curves. The lines $\{x = 0\}$ and $\{z = 0\}$ and the rational cubic $\{x^2z + xz^2 - 3xyz + y^3 = 0\}$. Notice that the rational cubic has a node at $[1 : 1 : 1]$. Moreover

$$d\omega = \left(\frac{dx}{x} + \frac{dz}{z} + \frac{2}{3}\frac{d(x^2z + xz^2 - 3xyz + y^3)}{x^2z + xz^2 - 3xyz + y^3} \right) \wedge \omega \,.$$

It can be verified that $\mathcal{F}$ has a projective structure in normal form with the polar divisor reduced and with support equal to the three $\mathcal{F}$-invariant curves. Thus the eccentricity of this projective structure is one.

4.4. **Proof of Proposition 2.** Let $\mathcal{F}$ be a quasi-minimal singular transversely projective foliation of $\mathbb{P}^2$ with transverse structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$ in normal form. If the monodromy of $\mathcal{H}$ is non-solvable and not minimal then there exists a non-algebraic proper closed set $\mathcal{M}$ of P formed by a union of leaves and singularities of $\mathcal{H}$.

If $L \subset \mathbb{P}^2$ is a generic line then $\mathrm{ecc}(\mathcal{P}) = -C^2$ where $C = \sigma(L)$. If $\mathrm{ecc}(\mathcal{P}) \leq 0$, i.e., $C^2 \geq 0$ then every leaf of $\mathcal{G}$, the restriction of $\mathcal{H}$ to $\pi^{-1}(L)$ must intersects $\mathcal{M} \cap \pi^{-1}(L)$. In the case $C^2 > 0$ this follows from [13, Corollary 8.2]. When $C^2 = 0$ we have that $\pi^{-1}(L) = \mathbb{P}^1 \times \mathbb{P}^1$ and every non algebraic leave must intersect every fiber of the *horizontal* fibration(otherwise the restriction of the second projection to it would be constant).

Therefore for L generic enough $\sigma^*\mathcal{M}$ is a non-algebraic proper closed subset of $\mathbb{P}^2$ invariant under $\mathcal{F}$. Thus $\mathcal{F}$ is not quasi-minimal. This contradiction implies the result. □

5. The Monodromy Representation

5.1. **A Local Obstruction.** Let $H = \{x_1 \cdot x_2 = 0\}$ be the union of the coordinate axis in $\mathbb{C}^2$ and $\rho : \pi_1(\mathbb{C}^2 \setminus H) \to \mathrm{PSL}(2, \mathbb{C})$ a representation.

Proposition 5.1. *If ρ is the monodromy representation of a transversely projective structure $\mathcal{P}$ defined in U, a neighborhood of $0 \in \mathbb{C}^2$, then ρ lifts to* $\mathrm{SL}(2, \mathbb{C})$.

Proof. We can suppose without loss of generality that U is a polydisc and that $\mathcal{P}$ is normal form. Over U every $\mathbb{P}^1$-bundle is trivial therefore $\mathcal{H}$ induces an integrable differential $\mathfrak{sl}(2,\mathbb{C})$-system on the trivial rank 2 vector bundle over U, cf. §2. Clearly ρ *lifts* to the monodromy of the $\mathrm{sl}(2,\mathbb{C})$-system and the proposition follows. □

A word of warning: it is not true that the monodromy of a transversely projective structure $\mathcal{P}$ always lift to $\mathrm{SL}(2,\mathbb{C})$. For instance we have smooth Riccati equations over elliptic curves with monodromy group conjugated to the abelian group

$$G =< (z_1 : z_2) \mapsto (z_2 : z_1); (z_1 : z_2) \mapsto (-z_1 : z_2) > .$$

5.2. **Prescribing the monodromy: Proof of Theorem 2.** First we will assume that H is an hypersurface with smooth irreducible components and with at most normal crossings singularities. Instead of working with the projective surface S we will work with a projective manifold M of arbitrary dimension n.

Construction of the $\mathbb{P}^1$-bundle and of the foliation. If $\rho : \pi_1(M \setminus H) \to \mathrm{SL}(2;\mathbb{C})$ is a representation then it follows from Deligne's work on Riemann-Hilbert problem [9] that there exists E, a rank 2 vector bundle over M, and a meromorphic flat connection

$$\nabla : E \to E \otimes \Omega^1_M(\log H)$$

with monodromy representation given by ρ. From the $\mathbb{C}$-linearity of ∇ we see that its solutions induce $\mathcal{H}$, a codimension one foliation of $\mathbb{P}(E)$. If $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) \to M$ denotes the natural projection then over $\pi^{-1}_{\mathbb{P}(E)}(M \setminus H)$ the restriction of $\mathcal{H}$ is nothing more than suspension of $[\rho] : \pi_1(M \setminus H) \to \mathrm{PSL}(2,\mathbb{C})$ as defined in [6, Example 2.8].

Let U be a sufficiently small open set of M and choose a trivialization of $E_{|U} = U \times \mathbb{C}^2$ with coordinates $(x, z_1, z_2) \in U \times \mathbb{C} \times \mathbb{C}$. Then for every section $\sigma = (\sigma_1, \sigma_2)$ of $E_{|U}$ we have that

$$\nabla_{|U}(\sigma) = \begin{pmatrix} d\sigma_1 \\ d\sigma_2 \end{pmatrix} + A \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is two by two matrix with $\alpha, \beta, \gamma, \delta \in \Omega^1_M(\log H)$ satisfying the integrability condition $dA + A \wedge A = 0$. Thus $\nabla = 0$ induces the system

$$\begin{aligned} dz_1 &= z_1\alpha + z_2\beta \\ dz_2 &= z_1\gamma + z_2\delta\,. \end{aligned}$$

Thus the solution of the above differential system are contained in the leaves of the foliation defined over $\pi^{-1}_{\mathbb{P}(E)}(U)$ by

$$\Omega_U = z_1 dz_2 - z_2 dz_1 - z_2^2\beta + z_1 z_2(\gamma - \alpha) + z_1^2\delta\,.$$

Clearly the foliations defined in this way patch together to give $\mathcal{H}$, a codimension one foliation on $\mathbb{P}(E)$ transverse to fibers of π which are not over H.

Construction of the meromorphic section. The next step in the proof of Theorem 2 is to assure the existence of a *generic* meromorphic section of $\mathbb{P}(E)$. This is done in the following

Lemma 5.2. *There exists a meromorphic section*

$$\sigma : M \dashrightarrow \mathbb{P}(E)\,,$$

with the following properties:

(i) *σ is generically transversal to $\mathcal{G}$;*
(ii) *$\overline{\mathrm{sing}(\sigma^*\Omega) \setminus (\sigma^*\mathrm{sing}(\mathcal{G}) \cup \mathrm{Ind}(\sigma))}$ has dimension zero.*

Proof. Let $\mathcal{L}$ be an ample line bundle over M. By Serre's Vanishing Theorem we have that for $k \gg 0$ the following properties holds:

(a) $E \otimes \mathcal{L}^k$ is generated by global sections;
(b) for every $x \in M$, $E \otimes \mathcal{L}^k \otimes m_x$ and $E \otimes \mathcal{L}^k \otimes m_x^2$ are also generated by global sections.

Using a variant of the arguments presented in [14, proposition 5.1] it is possible to settle that there exists a Zariski open $V \subset \mathrm{H}^0(M, E\otimes\mathcal{L}^{\otimes k})$ such that for every $s \in V$ the zeros locus of s is non-degenerated, of codimension two, with no irreducible component contained in the support of H and whose image does not contains any irreducible component of $\mathrm{sing}(\mathcal{F})$. We leave the details to the reader.

Let now $\mathcal{U} = \{U_i\}_{i\in I}$ be a finite covering of M by Zariski open subsets such that the restrictions of E and of the cotangent bundle of M to each U_i are both trivial bundles. For each $i \in I$ consider

$$\begin{array}{rcl} \Psi_i : U_i \setminus (U_i \cap H) \times \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k}) & \to & \mathbb{C}^n \\ (x, s) & \mapsto & s^*\Omega_i(x) \end{array}$$

where Ω_i is the 1-form over $\pi^{-1}_{\mathbb{P}(E)}(U_i)$ defining $\mathcal{G}|_{U_i}$ and $\Omega^1_{U_i}$ is implicitly identified with the trivial rank n vector bundle over U_i. It follows from (a) and (b) that for every $x \in M$ there exists sections in $\mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ with prescribed linear part at p. Thus if $Z_i = \Psi_i^{-1}(0)$ then

$$\dim Z_i = \mathrm{h}^0(M, E \otimes \mathcal{L}^{\otimes k}).$$

If $\rho_i : Z_i \to \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ is the natural projection then there exists a Zariski open set $W_i \subset \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ such that

$$\dim Z_i \leq \dim \rho^{-1}(s) + \mathrm{h}^0(M, E \otimes \mathcal{L}^{\otimes k}).$$

Thus $\dim \rho^{-1}(s) = 0$ for every $s \in W_i$.

A section $s \in \left(\bigcap_{i\in I} W_i\right) \cap V$ will induce a meromorphic section σ of $\mathbb{P}(E)$ with the required properties. □

Unicity. It remains to prove the unicity in the case that ρ is non-solvable. We will need the following

Lemma 5.3. *Suppose that $\pi : \mathbb{P}(E) \to M$ has a meromorphic section σ such that the foliation $\mathcal{F} = \sigma^*\mathcal{H}$ have non unique transversely projective structure. Then the monodromy representation of $\mathcal{H}$ is meta-abelian or there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^*\mathcal{R}$.*

Proof. After applying a fibered birational map we can assume that $\mathbb{P}(E) = M \times \mathbb{P}^1$ and that σ is the $[1 : 0]$-section, i.e., if

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \beta z_1 \cdot z_2 + \gamma z_2^2\,,$$

is the one form defining $\mathcal{H}$ then $\mathcal{F}$ is induced by α.

Since $\mathcal{F}$ has at least two non bimeromorphically equivalents projective structures then it follows from [15, proposition 2.1] (see also [6, lemma 2.20]) that there exists a rational function ℓ on M such that

$$d\alpha = -\frac{d\ell}{2\ell} \wedge \alpha \,.$$

Thus, after a suitable change of coordinates we can assume that $\beta = \frac{d\ell}{\ell}$. From the relation $d\beta = 2\alpha \wedge \gamma$ we deduce the existence of a rational function $f \in k(M)$ such that $\gamma = f\alpha$. Therefore $d\gamma = \beta \wedge \gamma$ implies that

$$\left(\frac{df}{f} - \frac{dl}{l}\right) \wedge \alpha = 0.$$

If $\mathcal{F}$ does not admit a rational first integral then $f = \ell$. Consequently, on the new coordinate system,

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \frac{d\ell}{2\ell} z_1 \cdot z_2 + \ell \alpha z_2^2 \,.$$

If $\Phi(x, [z_1 : z_2]) = (x, [z_1 : \sqrt{\ell} z_2])$ then we get

$$\frac{\Phi^*\Omega}{\sqrt{\ell}} = z_1 dz_2 - z_2 dz_1 + (z_1^2 + z_2^2)\frac{\alpha}{\sqrt{\ell}} \implies d\left(\frac{\Phi^*\Omega}{\sqrt{\ell}(z_1^2 + z_2^2)}\right) = 0,$$

meaning that after a ramified covering the foliation $\mathcal{H}$ is induced by a closed 1-form. Thus $\mathcal{H}$ has meta-abelian monodromy.

When $\mathcal{F}$ admits a rational first integral then it follows from [15, Theorem 4.1.(i)](see also [6, proposition 2.19]) that there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^* \mathcal{R}$. □

Back to the proof of Theorem 2 we apply lemma 5.2 to produce a section $\sigma : M \dashrightarrow \mathbb{P}(E)$ generically transversal to $\mathcal{H}$. If the transversely projective structure of $\mathcal{F} = \sigma^* \mathcal{H}$ is non unique then lemma 5.3 implies that there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^* \mathcal{R}$. Recall that we are assuming here that ρ is non-solvable.

As we saw in the proof of lemma 5.2 we have a lot of freedom when choosing σ. In particular we can suppose that $\phi \circ \sigma : M \dashrightarrow C \times \mathbb{P}^1$ is a dominant rational map. Thus $\mathcal{F}$ is the pull-back of Riccati foliation with non-solvable monodromy by a dominant rational map. The unicity of the transversely projectice structure of $\mathcal{F}$ follows from [15, proposition 2.1].

This is sufficient to conclude the proof of Theorem 2 under the additional assumption on H: normal crossing with smooth ireducible componentes. Notice that up to this point everything works for projective manifolds of arbitrary dimension.

To conclude we have just to consider the case where H is an arbitrary curve on a projective surface S. We can proceed as in the proof of Theorem 1, i.e., if we denote by $p : (\tilde{S}, \tilde{H} = p^* H) \to (S, H)$ the desingularization of H then there exists $\tilde{\rho} : \pi_1(\tilde{S}, \tilde{H}) \to \mathrm{SL}(2, \mathbb{C})$ such that $\rho = p_* \tilde{\rho}$. Thus we apply the previous arguments over $\tilde{S}$ and go back to S using lemma 3.2. □

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